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Dortmund, 14. April 2024

Report on the Habilitation Thesis of Ioannis Chrysikos

G-Structures, Dirac Operators with Torsion and Special Spinor Fields

In the realm of differential geometry and mathematical physics, G-structures and linear connections are fundamental concepts that provide a framework for understanding the geometry of smooth manifolds with additional geometric structures.

A G-structure on a manifold M is a principal G-bundle $P \to M$, where P is a collection of bases of the tangent spaces T_pM transformed into each other by an element of the group $G \subset Gl(n, \mathbb{R})$, n being the dimension of M. The most natural first order integrability invariant of a G-structure is the *intrinsic torsion*, which measures the failure of P to admit a *torsion free* connection.

For example, the intrinsic torsion of an $O(n)$-structure vanishes, since any Riemannian manifold admits a (unique) torsion free compatible connection. However, for most groups G the intrinsic torsion does not vanish, giving rise to *non-integrable geometries* whose study has become a tremendously growing field of research in recent decades.

In the case where $G \subset O(n)$ (which is the case mostly considered here), any G-structure on M induces a Riemannian metric g. We say that this structure has *parallel skew torsion*, if there is a connection ∇^c compatible with P whose torsion $\mathrm{Tor} \in \Omega^2(M, TM)$ is encoded as

$$T(x, y, z) = g(\mathrm{Tor}(x, y), z)$$

for a ∇^c-parallel 3-form T. If P admits such a connection ∇^c, then it is unique and called the *canonical connection* of the structure. Examples of non-integrable structures with parallel skew torsion are Sasakian and 3-Sasakian manifolds, nearly Kähler manifolds, and nearly parallel G_2-manifolds.

For a geometry with parallel skew torsion, one may consider the pencil of connections ∇^s for $s \in \mathbb{R}$ connecting the canonical and the Levi-Civita connection. Assuming that M is spin, each connection ∇^s induces a Dirac operator D^s on spinors. The square $(D^s)^2$ is then an elliptic self-adjoint differential operator, whose symbol coincides with that of the Lichnerowicz-Laplacian, so that an estimate of the eigenvalues will be important. This is the field of the first part of this Habilitation thesis.

The second series of papers in this Habilitation deals with the study of *homogeneous non-integrable geometries.* That is, one considers L-structures on a homogeneous space G/K, where the action of G extends to the principal L-bundle $P \to G/K$. For this, classification results as well as the construction of new examples of structures with skew parallel torsion for different choices of structure groups L are provided.

In the following, I will briefly summarize the contributions of the five publications submitted for this habilitation.

A *Killing and twistor spinors with torsion*

In this paper, Chrysikos defines *Killing and twistor spinors* for spin manifolds with torsion along the lines explained above, and investigates their "classical" properties. The pencil of connections ∇^s, $s \in \mathbb{R}$ is given by

$$\langle \nabla^s_X Y, Z\rangle = \langle \nabla^g_X Y, Z\rangle + 2sT(X, Y, Z),$$

where ∇^g denotes the Levi-Civita connection. The torsion form of ∇^s is $4sT$ for the fixed 3-form T from before. While $s = 0$ corresponds to the Levi-Civita connection and is thus torsion free, the values $s = 1/4$ corresponds to the canonical connection ∇^c. As it turns out, the value $s = 1/12$ is also special, and $\nabla^{1/12}$ is called the *cubic connection.*

Killing spinors and twistor spinors, respectively, are defined by the identities

$$\nabla^s_X \psi = \lambda X \cdot \psi \qquad \text{and} \qquad \nabla^s_X \psi = \frac{1}{n} X \cdot D^s \psi, \qquad \text{respectively,}$$

and assuming that $\nabla^c T = 0$, Chrysikos interprets the twistor equation as a parallelity condition and thus shows that twistor spinors have isolated zeros. Furthermore, if there is a ∇^c-parallel twistor spinor with torsion which is also a T-eigenspinor, then M is both Einstein and ∇^c-Einstein. For ∇^c-parallel spinors, he also points out a correspondence of the Killing spinor equation with torsion given above with the usual equation for Killing spinors.

He then gives a number of classes of examples: for instance, he considers naturally reductive homogeneous spaces for which the characteristic connection is the canonical (Maurer-Cartan) connection. He gives examples of Killing spinors with torsion on nearly Kähler or nearly parallel G_2-manifolds in dimensions 6 and 7, respectively; and finally on the standard sphere S^3, considered as an example of a spin manifold with torsion.

B *A new $\frac{1}{2}$-Ricci type formula on the spinor bundle and applications*

Here, Chrysikos extends several results known in (torsion free) Spin geometry to the case where the connection is given by ∇^s. In particular, he investigates the Ricci tensor Ric^s of ∇^s. Again, assuming that T is parallel under the canonical connection $\nabla^c = \nabla^{1/4}$, he shows that the action of the Ricci-endomorphism $\mathrm{Ric}^s(X)$ on the spinor bundle $\Sigma^g M$ can be described in terms of the Dirac operator $\not{D}^s$ by an explicit formula that extends a formula given earlier by Kim and Friedrich in the classical case.

There is another result of Friedrich that he generalizes. Friedrich showed that the $\frac{1}{2}$-Ric^g-identity induces the Schrödinger-Lichnerowicz formula to the Dirac operator D^g, thus relating the square $(D^g)^2$ to the Laplacian $(\nabla^g)^*\nabla^g$. As a generalization, Chrysikos shows that – again under the assumtion that T is parallel w.r.t. $\nabla^c = \nabla^{1/4}$ – the analogous formula for $\frac{1}{2}$-Ric^s holds as well.

As an application, he shows how this $\frac{1}{2}$-Ric^s-identity can be used for the study of ∇^s-parallel and ∇^c-parallel spinors for certain classes of manifolds, such as 5-dimensional Sasakian manifolds, 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel G_2-manifolds.

C *Invariant connections and ∇-Einstein structures on isotropy irreducible spaces.*

In this article, co-authored with Gustad and Winther, the authors consider G-invariant connections on an isotropy irreducible homogeneous space $M = G/K$. Such connections are in bijective correspondence with linear maps $\mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the structure group, satisfying certain conditions.

As a consequence of their classification, they determine the dimension of the space of invariant connections for each compact isotropy irreducible non-symmetric homogeneous space. Then they consider the parallelity of the torsion tensor and determine the dimension of the space of connections with parallel skew torsion in all cases. Moreover, they also give a classification of G-invariant ∇-Einstein structures on effective non-symmetric strongly isotropy irreducible spaces.

D *Homogeneous 8-manifolds admitting invariant $Spin(7)$-structures*

This article is joint work with Alekseevski, Fino and Raffero, and considers homogeneous spaces admitting an invariant $Spin(8)$-structure. That is, just as in the preceding paper, one considers an 8-dimensional homogeneous space G/H for compact groups $H \subset G$, such that the tangent space admits an H-invariant 4-form Φ which determines a $Spin(7)$-structure. These structures cannot be torsion free, as torsion free $Spin(7)$-structures are Ricci flat, and by a classical result of Alekseevski, homogeneous Ricci flat spaces must be flat.

There is a description of the intrinsic torsion of $Spin(7)$-manifolds by Fernandez, and the authors also describe the homogeneous $Spin(7)$-structures of their classification, determining which types of the Fernandez classification of these torsions can be assumed in each case, and in which of these cases the torsion is skew.

E *Differential geometry of $SO^*(2n)$-structures*

This article is co-authored with Gregorovič and Winther. The Lie group $SO^*(2n)$ is a non-compact real form of the complex Lie group $SO(2n,\mathbb{C})$. It can be described as the invariance groups of the (unique) quaternionically skew-Hermitean form on the quaternionic vector space $\mathbb{H}^n$. Adding the scalar multiplication of unit quaternions, one may extend this to $SO^*(2n) \cdot Sp(1) \subset Gl(4n,\mathbb{R})$.

A G-structure with one of these groups on a $4n$-dimensional manifold is called *almost hypercomplex skew hermitean structure* and *almost qiaternionic skew Hermitean structure*, respectively. The aim of the paper is to systematically investigate $4n$-dimensional manifolds with either of these structures.

The most natural description is to describe these structures as a combination of an amost symplectic structure plus a hypercomplex (hyperquaternionic, respectively) structure. This observation allows to determine the G-invariant decomposition of the exterior algebra for either of these two groups.

They also observe that there is a canonical symmetric four-tensor whose invariance group is $SO^*(2n) \cdot Sp(1)$; this 4-tensor can be thought of as an analogue of the symmetric 4-form determining a quaternionic structure (i.e., a geometry with structure group $Sp(n)Sp(1)$).

With these algebraic prerequisits at hand, the authors study the intrinsic torsion of these two structures in question, getting a Gray-type characterization of the various torsion types. The decomposition of the torsion is rather complicated, leading to a large number of possible torsion types. Besides discussing the most natural ones, the authors also provide examples of torsion free examples with either of these structure groups.

Evaluation of the Habilitation

This habilitation represents a remarkable contribution to mathematics. Through this collection of papers, Dr. Chrysikos showcases his significant impact on contemporary research in non-integrable geometries. Papers **A** and **B**, authored solely by him, illustrate his adeptness in advancing the study of non-integrable geometries, a pursuit pioneered by Thomas Friedrich and his academic cohort in various geometrical contexts. Dr. Chrysikos adeptly reveals the parallels in phenomena within geometries featuring parallel torsion and skillfully characterizes their fundamental properties.

The remaining three papers, **C**, **D** and **E**, authored in collaboration with colleagues, leverage algebraic methodologies to explore geometries invariant under symmetry groups. This approach necessitates the utilization of sophisticated techniques from the theory of semi-simple Lie algebras, often entailing formidable technical challenges. I am profoundly impressed by Dr. Chrysikos and his co-authors' adept navigation of these obstacles, resulting in classification outcomes that offer noteworthy instances of Einstein structures, homogeneous $Spin(7)$ structures and $SO^*(2n)$ geometries.

Beyond the works highlighted in this habilitation, Dr. Chrysikos possesses a repertoire of noteworthy publications that have garnered significant recognition within the scientific community, securing publication in esteemed journals.

Given this impressive body of work, I wholeheartedly recommend the acceptance of Dr. Chrysikos' Habilitation thesis and his subsequent awarding of Habilitation.

(Prof. Dr. Lorenz Schwachhöfer)